Reverse Rearrangement Inequality

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Edited by Cody Johnson

November 27, 2014

Abstract

My name is **Daniel Liu**[1] and I am a high school freshman. In this paper I will be presenting and proving a new inequality I found, which I dub the "**Reverse Rearrangement Inequality**". I named it this because of its striking similarity to the well-known Rearrangement Inequality, although its resemblance was purely coincidental: I happened upon it when creating a problem for the Proofathon competition. This inequality specializes in problems with products on both sides, and on problems with permutated sequences.

A very special thanks to **Cody Johnson**[2] for properly formatting this PDF and editing; without him, I would have never been able to even begin creating this paper. Also, a shoutout to Art of Problem Solving users *jh235*, *vincenthuang75025*, *minimario*, *FlakeLCR*, *mathtastic*, *mursalinmath* and any others who I may have forgotten to mention for helping me out along the way.

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1 Definitions and Notation

Let $\{a_1, a_2, \dots a_n\}$ and $\{b_1, b_2, \dots b_n\}$ be two non-negative sequences of real numbers. For example, $a_1 = 1$, $a_2 = 5$, $a_3 = \pi$, $a_4 = 0$, $a_5 = 9001$ is a valid sequence. However, $a_1 = 4, a_2 = -3, a_3 = i + 1$ is not a valid sequence because it contains a negative real, and it contains a non-real number.

Two sequences $\{a\}$ and $\{b\}$ are said to be *similarly ordered* if $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$, or $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$.Two sequences $\{a\}$ and $\{b\}$ are said to be *oppositely ordered* if $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, or $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \le b_2 \le \cdots \le b_n$.

$\sigma(1), \sigma(2), \dots \sigma(n)$ denotes a permutation of the numbers $1, 2, \dots n$. For example, $(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) = (1, 4, 3, 2)$ is a valid permutation. $(1, 4, 2, 2)$ is not a valid permutation because both $\sigma(3)$ and $\sigma(4)$ take on the value of 2. Note: $(1, 2, 3, 4)$ is a valid permutation; it is known as the identity permutation.

The permutation $\sigma(1), \sigma(2), \dots \sigma(n)$ will sometimes be referred to simply as σ. Thus, for example, the permutation σ' would be referencing the permutation $\sigma'(1), \sigma'(2), \dots \sigma'(n)$

2 Reverse Rearrangement Inequality

Theorem. Given two sequences $\{a\}$ and $\{b\}$ that are similarly ordered, the inequality

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n) \le (a_1 + b_{\sigma(1)})(a_2 + b_{\sigma(2)}) \cdots (a_n + b_{\sigma(n)})$$

is true.

If instead, $\{a\}$ and $\{b\}$ are oppositely ordered, then the inequality

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n) \ge (a_1 + b_{\sigma(1)})(a_2 + b_{\sigma(2)}) \cdots (a_n + b_{\sigma(n)})$$

is true.

These two inequalities can be combined and compactly written as

$$\prod_{k=1}^{n} (a_k + b_k) \le \prod_{k=1}^{n} (a_k + b_{\sigma(k)}) \le \prod_{k=1}^{n} (a_k + b_{n-k+1})$$

given that $\{a\}$ and $\{b\}$ are similarly ordered.

Example.

As an explicit example, when $n = 3$, and $(\sigma(1), \sigma(2), \sigma(3)) = (2, 3, 1)$, we get the inequality

$$(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) \le (a_1 + b_2)(a_2 + b_3)(a_3 + b_1) \le (a_1 + b_3)(a_2 + b_2)(a_3 + b_1)$$

2.1 Proof of Lower Bound

Suppose that $\{a\}$ and $\{b\}$ are similarly ordered; we want to prove that

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n) \le (a_1 + b_{\sigma(1)})(a_2 + b_{\sigma(2)}) \cdots (a_n + b_{\sigma(n)})$$

for all permutations σ.

We shall proceed with induction on n.

Base Case. $n = 1$.

Clearly $a_1 + b_1 \le a_1 + b_1$.

Induction Hypothesis. Assume that

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_k + b_k) \le (a_1 + b_{\sigma(1)})(a_2 + b_{\sigma(2)}) \cdots (a_k + b_{\sigma(k)})$$

for all permutations σ.

We want to prove that

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_{k+1} + b_{k+1}) \le (a_1 + b_{\sigma(1)})(a_2 + b_{\sigma(2)}) \cdots (a_{k+1} + b_{\sigma(k+1)})$$

for all permutations σ.

Let i, j be the unique pair of positive integers (possibly equal) such that $\sigma(i) = k + 1$ and $\sigma(k + 1) = j$.

Now define a unique permutation $\sigma'(1), \sigma'(2), \ldots \sigma'(k)$ of $1, 2, \ldots k$ such that $\sigma'(x) = \sigma(x)$ for all $x \neq i$, and $\sigma'(i) = j$. This permutation can always be formed because we are simply reducing $\sigma(i) = k+1 \to \sigma(k+1) = j$ to $\sigma'(i) = j$.

We want to prove that

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_{k+1} + b_{k+1}) \leq (a_1 + b_{\sigma(1)}) \cdots (a_i + b_{\sigma(i)}) \cdots (a_{k+1} + b_{\sigma(k+1)})$$

Using our induction hypothesis with the permutation σ', we see that

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_{k+1} + b_{k+1}) \leq (a_1 + b_{\sigma'(1)})(a_2 + b_{\sigma'(2)}) \cdots (a_k + b_{\sigma'(k)})(a_{k+1} + b_{k+1})$$

By the definition of σ', we can substitute $\sigma(x)$ in for $\sigma'(x)$ for all $x \neq i$ and j in for $\sigma'(i)$:

$$(a_1 + b_{\sigma'(1)}) \cdots (a_k + b_{\sigma'(k)})(a_{k+1} + b_{k+1}) = (a_1 + b_{\sigma(1)}) \cdots (a_i + b_j) \cdots (a_k + b_{\sigma(k)})(a_{k+1} + b_{k+1})$$

Now for a lemma:

Lemma.

$$(a_i + b_j)(a_{k+1} + b_{k+1}) \leq (a_i + b_{k+1})(a_{k+1} + b_j)$$

Proof.

Expanding gives

$$a_i a_{k+1} + a_i b_{k+1} + b_j a_{k+1} + b_j b_{k+1} \leq a_{k+1} b_{k+1} + a_i a_{k+1} + b_j b_{k+1} + a_i b_j$$

Simplifying:

$$a_{k+1} b_{k+1} + a_i b_j - a_i b_{k+1} - b_j a_{k+1} \geq 0$$

This factors as

$$(a_{k+1} - a_i)(b_{k+1} - b_j) \geq 0$$

Since $\{a\}$ and $\{b\}$ are similarly ordered, either both $a_{k+1} - a_i$ and $b_{k+1} - b_j$ are positive or both are negative or, in the case of $\sigma(k+1) = k+1$, both are zero; either way, the inequality is true. Thus the lemma is true. ■

Using this lemma, we see that

$$(a_1 + b_{\sigma(1)}) \cdots (a_i + b_j) \cdots (a_k + b_{\sigma(k)})(a_{k+1} + b_{k+1}) \leq (a_1 + b_{\sigma(1)}) \cdots (a_i + b_{k+1}) \cdots (a_k + b_{\sigma(k)})(a_{k+1} + b_j)$$

But wait! $k + 1 = \sigma(i)$ and $j = \sigma(k + 1)$, so

$$(a_1 + b_{\sigma(1)}) \cdots (a_i + b_{k+1}) \cdots (a_k + b_{\sigma(k)})(a_{k+1} + b_j) = (a_1 + b_{\sigma(1)})(a_2 + b_{\sigma(2)}) \cdots (a_{k+1} + b_{\sigma(k+1)})$$

Therefore

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_{k+1} + b_{k+1}) \leq (a_1 + b_{\sigma(1)})(a_2 + b_{\sigma(2)}) \cdots (a_{k+1} + b_{\sigma(k+1)})$$

Note that this proof works for any arbitrary permutation σ, so we are done. □

Summary.

$$\begin{aligned}
(a_1 + b_1)(a_2 + b_2) \cdots (a_{k+1} + b_{k+1}) &\leq (a_1 + b_{\sigma'(1)})(a_2 + b_{\sigma'(2)}) \cdots (a_k + b_{\sigma'(k)})(a_{k+1} + b_{k+1}) && \text{(ind. hyp.)} \\
&= (a_1 + b_{\sigma(1)}) \cdots (a_i + b_j) \cdots (a_k + b_{\sigma(k)})(a_{k+1} + b_{k+1}) && \text{(def. of } \sigma'\text{)} \\
&\leq (a_1 + b_{\sigma(1)}) \cdots (a_i + b_{k+1}) \cdots (a_k + b_{\sigma(k)})(a_{k+1} + b_j) && \text{(lemma)} \\
&= (a_1 + b_{\sigma(1)})(a_2 + b_{\sigma(2)}) \cdots (a_{k+1} + b_{\sigma(k+1)}) && \text{(def. of } \sigma\text{)}
\end{aligned}$$

In the induction step above equality happens when $\sigma(k + 1) = k + 1$. This means the equality case for Reverse Rearrangement is when $\sigma(k) = k$ for all $k = 1 \to n$.

2.2 Proof of Upper Bound

Proving the maximum inequality is extremely similar to proving the minimum inequality. The proof will be left as an exercise to the reader.

2.3 Corollaries

We can also create two corollaries:

Corollary 1a.

Given a decreasing function $f(x)$ such that $f(x) \geq 0$ for all non-negative reals x, and non-negative reals $a_1, a_2, \ldots a_n$,

$$(a_1 + f(a_1))(a_2 + f(a_2)) \cdots (a_n + f(a_n)) \geq (a_1 + f(a_{\sigma(1)}))(a_2 + f(a_{\sigma(2)})) \cdots (a_n + f(a_{\sigma(n)}))$$

This can be compactly written as

$$\prod_{k=1}^{n} (a_k + f(a_k)) \geq \prod_{k=1}^{n} (a_k + f(a_{\sigma(k)}))$$

Proof.

Assume WLOG that $a_1 \leq a_2 \leq \cdots \leq a_n$. Let $f(a_i) = b_i$ for all $i = 1 \to n$. Note that $\{a\}$ and $\{b\}$ are oppositely ordered because $f(x)$ is decreasing. By the Reverse Rearrangement Inequality, we are done. □

Corollary 1b.

Given an increasing function $f(x)$ such that $f(x) \geq 0$ for all non-negative reals x, and non-negative reals $a_1, a_2, \ldots a_n$,

$$(a_1 + f(a_1))(a_2 + f(a_2)) \cdots (a_n + f(a_n)) \leq (a_1 + f(a_{\sigma(1)}))(a_2 + f(a_{\sigma(2)})) \cdots (a_n + f(a_{\sigma(n)}))$$

This can be compactly written as

$$\prod_{k=1}^{n} (a_k + f(a_k)) \leq \prod_{k=1}^{n} (a_k + f(a_{\sigma(k)}))$$

Proof.

Assume WLOG that $a_1 \leq a_2 \leq \cdots \leq a_n$. Let $f(a_i) = b_i$ for all $i = 1 \to n$. Note that $\{a\}$ and $\{b\}$ are similarly ordered because $f(x)$ is increasing. By the Reverse Rearrangement Inequality, we are done. □

Corollary 2a.

Given a decreasing function $f(x)$ such that $f(x) > 0$ for all non-negative reals x, and non-negative reals $a_1, a_2, \ldots a_n$,

$$(a_1 f(a_1) + 1)(a_2 f(a_2) + 1) \cdots (a_n f(a_n) + 1) \leq (a_1 f(a_{\sigma(1)}) + 1)(a_2 f(a_{\sigma(2)}) + 1) \cdots (a_n f(a_{\sigma(n)}) + 1)$$

This can be compactly written as

$$\prod_{k=1}^{n} (a_k f(a_k) + 1) \leq \prod_{k=1}^{n} (a_k f(a_{\sigma(k)}) + 1)$$

Proof.

Let $g(x) = \dfrac{1}{f(x)}$ where $f(x)$ is an increasing function. Applying Corollary 1b with $g(x)$ gives

$$\left(a_1 + \frac{1}{f(a_1)}\right)\left(a_2 + \frac{1}{f(a_2)}\right) \cdots \left(a_n + \frac{1}{f(a_n)}\right) \leq \left(a_1 + \frac{1}{f(a_{\sigma(1)})}\right)\left(a_2 + \frac{1}{f(a_{\sigma(2)})}\right) \cdots \left(a_n + \frac{1}{f(a_{\sigma(n)})}\right)$$

Multiplying both sides by $f(a_1)f(a_2)\cdots f(a_n)$ gives

$$(a_1f(a_1)+1)(a_2f(a_2)+1)\cdots(a_nf(a_n)+1) \le (a_1f(a_{\sigma(1)})+1)(a_2f(a_{\sigma(2)})+1)\cdots(a_nf(a_{\sigma(n)})+1)$$

as desired.

Corollary 2b.

Given an increasing function $f(x)$ such that $f(x)>0$ for all non-negative reals x, and non-negative reals $a_1, a_2, \dots a_n$,

$$(a_1f(a_1)+1)(a_2f(a_2)+1)\cdots(a_nf(a_n)+1) \ge (a_1f(a_{\sigma(1)})+1)(a_2f(a_{\sigma(2)})+1)\cdots(a_nf(a_{\sigma(n)})+1)$$

This can be compactly written as

$$\prod_{k=1}^{n}(a_kf(a_k)+1) \ge \prod_{k=1}^{n}(a_kf(a_{\sigma(k)})+1)$$

Proof.

Let $g(x)=\dfrac{1}{f(x)}$ where $f(x)$ is a decreasing function. Applying Corollary 1a with $g(x)$ gives

$$\left(a_1+\frac{1}{f(a_1)}\right)\left(a_2+\frac{1}{f(a_2)}\right)\cdots\left(a_n+\frac{1}{f(a_n)}\right) \ge \left(a_1+\frac{1}{f(a_{\sigma(1)})}\right)\left(a_2+\frac{1}{f(a_{\sigma(2)})}\right)\cdots\left(a_n+\frac{1}{f(a_{\sigma(n)})}\right)$$

Multiplying both sides by $f(a_1)f(a_2)\cdots f(a_n)$ gives

$$(a_1f(a_1)+1)(a_2f(a_2)+1)\cdots(a_nf(a_n)+1) \ge (a_1f(a_{\sigma(1)})+1)(a_2f(a_{\sigma(2)})+1)\cdots(a_nf(a_{\sigma(n)})+1)$$

as desired.

3 Example Problems

1. For all positive integers n, prove that

$$n! \leq \left(\frac{n+1}{2}\right)^n$$

Proof.

First, we see that we have products on both sides of the inequality, which tells us that Reverse Rearrangement might work. In particular, it seems that the two sequences $\{a\}$ and $\{b\}$ perfectly pair up with each other in order to create the RHS. Indeed, if we let $a_k = b_k = \frac{k}{2}$ for $k = 1 \to n$, they add up, when similarly ordered, to $1, 2, \ldots n$ and, when oppositely ordered, $\frac{n+1}{2}, \frac{n+1}{2}, \ldots \frac{n+1}{2}$. Thus, by Reverse Rearrangement, we have

$$1 \cdot 2 \cdots n \leq \left(\frac{n+1}{2}\right)^n$$

so we are done. □

2. Let x, y, z be positive reals. Prove that

$$\left(\frac{x}{y} + \frac{y^2}{x}\right)\left(\frac{y}{z} + \frac{z^2}{y}\right)\left(\frac{z}{x} + \frac{x^2}{z}\right) \geq (x+1)(y+1)(z+1)$$

Proof.

Let's first add the fractions on the LHS to get

$$\left(\frac{x^2+y^3}{xy}\right)\left(\frac{y^2+z^3}{yz}\right)\left(\frac{z^2+x^3}{zx}\right) \geq (x+1)(y+1)(z+1)$$

Multiplying both sides by $x^2y^2z^2$ gives

$$(x^2+y^3)(y^2+z^3)(z^2+x^3) \geq x^2y^2z^2(x+1)(y+1)(z+1)$$

In order to use Reverse Rearrangement, we want to create something like $x^2 + x^3$ on the RHS. Fortunately, this is easy: just combine the x^2 with the $x + 1$ and ditto for y, z.

$$(x^2+y^3)(y^2+z^3)(z^2+x^3) \geq (x^3+x^2)(y^3+y^2)(z^3+z^2)$$

Let $a_1 = x^2, a_2 = y^2, a_3 = z^2$, and $b_1 = x^3, b_2 = y^3, b_3 = z^3$. We see that $\{a\}$ and $\{b\}$ are similarly ordered, so we can use Reverse Rearrangement to prove this inequality. □

3. Let $a_1, a_2, \ldots a_n$ be non-negative reals. Prove that

$$\left(\frac{1}{a_1} + a_2 + 2\right)\left(\frac{1}{a_2} + a_3 + 2\right)\cdots\left(\frac{1}{a_n} + a_1 + 2\right) \leq \frac{(a_1+1)^2(a_2+1)^2\cdots(a_n+1)^2}{a_1a_2\cdots a_n}$$

Proof.

We see that we have a bunch of products, which seems promising. However, the LHS has three terms, while our inequality only has two terms. We can divide the three terms into two terms, but it's currently not that easy to decide what to divide. Thus, let's look at the RHS first. Expanding the RHS gives

$$RHS = \frac{(a_1^2+2a_1+1)(a_2^2+2a_2+1)\cdots(a_n^2+2a_n+1)}{a_1a_2\cdots a_n} = \left(\frac{a_1^2+2a_1+1}{a_1}\right)\left(\frac{a_2^2+2a_2+1}{a_2}\right)\cdots\left(\frac{a_n^2+2a_n+1}{a_n}\right)$$

But note that

$$\frac{(a_i^2 + 2a_i + 1)}{a_i} = a_i + 2 + \frac{1}{a_i}$$

Now it's very clear what we want to divide the terms of the LHS into: Let $f(a_i) = \dfrac{1}{a_i} + 2$. This function is decreasing, so we can use Corollary 1a:

$$(a_1 + f(a_1))(a_2 + f(a_2)) \cdots (a_n + f(a_n)) \geq (a_1 + f(a_n))(a_2 + f(a_1)) \cdots (a_n + f(a_{n-1}))$$

Plugging $f(x) = \dfrac{1}{x} + 2$ back in, we get

$$\left(a_1 + \frac{1}{a_1} + 2\right)\left(a_2 + \frac{1}{a_2} + 2\right) \cdots \left(a_n + \frac{1}{a_n} + 2\right) \geq \left(a_1 + \frac{1}{a_n} + 2\right)\left(a_2 + \frac{1}{a_1} + 2\right) \cdots \left(a_n + \frac{1}{a_{n-1}} + 2\right)$$

Note that we found earlier that $a_i + \dfrac{1}{a_i} + 2 = \dfrac{(a_i + 1)^2}{a_i}$. Subbing that back in gives us

$$\frac{(a_1 + 1)^2(a_2 + 1)^2 \cdots (a_n + 1)^2}{a_1 a_2 \cdots a_n} \geq \left(a_1 + \frac{1}{a_n} + 2\right)\left(a_2 + \frac{1}{a_1} + 2\right) \cdots \left(a_n + \frac{1}{a_{n-1}} + 2\right)$$

But this is our desired result, so we are done. □

4. Let x, y, z be non-negative real numbers. Prove that

$$(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) \geq \left(\frac{x^2 y + 1}{y + 1}\right)\left(\frac{y^2 z + 1}{z + 1}\right)\left(\frac{z^2 x + 1}{x + 1}\right)$$

Proof.

First, multiply both sides by $(x + 1)(y + 1)(z + 1)$. This reduced the LHS to $(x^3 + 1)(y^3 + 1)(z^3 + 1)$ so we just need to prove that

$$(x^3 + 1)(y^3 + 1)(z^3 + 1) \geq (x^2 y + 1)(y^2 z + 1)(z^2 x + 1)$$

This looks like a nice time to use Corollary 2b, with the function $f(x) = x^2$. Proceeding with that path:

$$(xf(x) + 1)(yf(y) + 1)(zf(z) + 1) \geq (yf(x) + 1)(zf(y) + 1)(xf(z) + 1)$$

which is true by Corollary 2b. □

4 Important Concepts

Before starting the problems section here are some important concepts to successfully using the Reverse Rearrangement Inequality. I recommend looking over this list before and after solving a problem.

1. *Look for products on both sides of the inequality.*

   If you see products on both sides of the inequality, then it should trigger your brain to put a mental note that Reverse Rearrangement might be applicable. Reverse Rearrangement usually cannot handle inequalities with terms being added together on either side, as it is hard to turn that into a product.

2. *Look for permutations.*

   If you see that the two sides of the inequality are essentially the same except for a permutation of a group of variables, then Reverse Rearrangement may be an option. An extremely common permutation to look out for is

   $$\sigma(a_1) = a_2, \sigma(a_2) = a_3, \ldots \sigma(a_n) = a_1$$

3. *Look for problems having only one variable per term.*

   If both sides of the inequality have terms that have more than one variable, try to transform one of the sides to have only one variable per term.

   As an example, the expression

   $$(x^2 + y + 1)(y^2 + z + 1)(z^2 + x + 1)$$

   has more than one variable per term but

   $$(x^2 + x + 1)(y^2 + y + 1)(z^2 + z + 1)$$

   has one variable per term.

4. *Try to make corresponding terms have the same degree.*

   If the corresponding terms in each pair of terms is the same degree or you can make it the same degree, then it will help in successfully applying the Reverse Rearrangement Inequality. For example, consider

   $$(ab + bcd + d) \iff (c^2 + a^3 + b)$$

   we can see that the degrees of ab and c^2 are both 2, the degrees of bcd and a^3 are both 3, and the degrees of d and b are both 1.

5. *If no clear way to permutate the variables is seen, try a substitution.*

   Substituting often takes an expression that Reverse Rearrangement can't handle to an expression it can handle. Substituting can also help get rid of a variable that Reverse Rearrangement needs to get rid of in order to work properly.

   For example, given the expression $(a + b + c)$ it is troublesome to use Reverse Rearrangement due to its three variables. However, we can transform it into

   $$a\left(1 + \frac{b}{a} + \frac{c}{a}\right) = a\left(1 + \frac{1}{x} + z\right)$$

   where $x = \frac{a}{b}$, $y = \frac{b}{c}$, and $z = \frac{c}{a}$, and after getting rid of the a by cancellation from both sides or another method, we only have two variables per term, ideal for Reverse Rearrangement.

6. ***Use Algebraic Manipulations.***

   Algebraic manipulation is a cornerstone of successfully using Reverse Rearrangement. Spotting the right algebraic manipulation to use in a given situation will greatly increase your chances of solving a Reverse Rearrangement problem.

5 Problems

5.1 Problems

1. Let $\{a_1, a_2, \dots a_n\}$ be a permutation of $\{1, 2, \dots n\}$.

   Prove that

$$\prod_{k=1}^{n} (a_k + k) \leq (n+1)^n$$

2. Let $a_1, a_2, \dots a_n$ and $b_1, b_2, \dots b_n$ be two non-negative increasing sequences.

   Prove that

$$\prod_{i=1}^{n} \prod_{j=1}^{n} (a_i + b_j) \geq \prod_{k=1}^{n} (a_k + b_k)^n$$

3. Given that $x, y \geq 1$ where x, y are reals prove that

$$(x^2 - x + y)(y^2 - y + x) \geq x^2 y^2$$

4. Given that $a_1, a_2, \dots a_n \geq 1$ are reals, prove that

$$(a_1^2 - a_1 + a_2)(a_2^2 - a_2 + a_3) \cdots (a_n^2 - a_n + a_1) \geq a_1^2 a_2^2 \cdots a_n^2$$

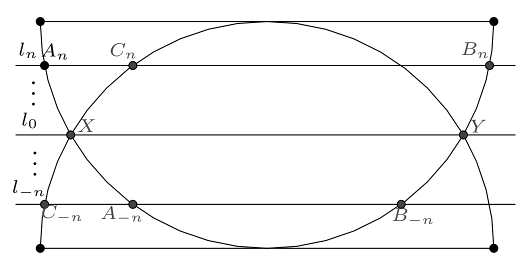
5. Define reals $x, y, z \geq 1$. Prove that

$$\ln ex^{ex} \ln ey^{ey} \ln ez^{ez} \geq \ln ex^{ey} \ln ey^{ez} \ln ez^{ex}$$

6. Given non-negative reals $k, a_1, a_2, \dots a_n$, prove that

$$\prod_{cyc} \left(k^{a_1 - a_2} + 1\right) \geq 2^n$$

7. Two half-circles are drawn as shown below, with a line l_0 through the two intersections points, X, Y of the half-circles as shown. Lines l_k for $k = -n \to n$ parallel to the bases of the half-circles are drawn such that the distances between l_k and l_0, and l_{-k} and l_0 are always the same for all $k = 1 \to n$.



   The intersection points of l_k with one of the half-circles are labeled A_k, B_k, and with the other half-circle at C_k, as shown in the diagram.

   Prove that

$$\prod_{k=-n}^{n} |A_k B_k| \leq \prod_{k=-n}^{n} |B_k C_k|$$

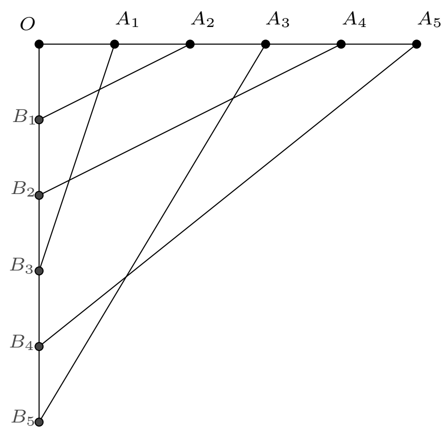
8. Given that $a_1, a_2, \dots a_n$ are non-negative real numbers and $k \geq 1$ is a positive integer, prove that

$$(a_1^2 + 2)(a_2^2 + 2) \cdots (a_n^2 + 2) \geq (a_1 a_2 + 2)(a_2 a_3 + 2) \cdots (a_n a_1 + 2)$$

9. Given positive negative reals $k, a_1, a_2, \dots a_n$ satisfying $a_1 a_2 \cdots a_n = 1$, prove that

$$(a_1 + k)(a_2 + k) \cdots (a_n + k) \geq (k+1)^n$$

10. Points $A_1, \dots A_5$ are collinear with O, and so are points $B_1, \dots B_5$, with $\angle A_1 O B_1 = 90°$. In addition, the distance between adjacent points is 1, as shown in the diagram.



Points $A_1, \dots A_5$ are connected with one of $B_1, \dots B_5$ each at random; this creates 5 lines having lengths of $L_1, \dots L_5$.

Prove that

$$480\sqrt{2} \leq L_1 L_2 L_3 L_4 L_5 \leq 1560\sqrt{2}$$

5.2 Challenge Problems

1. Given that $a_1, a_2, \dots a_n$ are non-negative real numbers, prove that

$$(a_1^2 + 2a_1)(a_2^2 + 2a_2) \cdots (a_n^2 + 2a_n) \leq (a_1 a_2 + a_1 + a_2)(a_2 a_3 + a_2 + a_3) \cdots (a_n a_1 + a_n + a_1)$$

2. Given that $0 \leq \alpha, \beta, \gamma \leq \pi$, prove that

$$\prod_{cyc} (3 \sin \alpha - \sin 3\alpha) \leq \prod_{cyc} (\sin^3 \alpha + 3 \sin \alpha \sin \beta \sin \gamma)$$

3. Given that x, y, z are non-negative reals such that $xy + yz + zx = 1$, prove that

$$(x^2 + y^2 + 2)(y^2 + z^2 + 2)(z^2 + x^2 + 2) \geq 8(x + y + z - xyz)^2$$

4. Given that $a_1, a_2, \dots a_n$ are non-negative real numbers, prove that

$$\prod_{cyc} (a_1^3 + a_2 + 1) \geq \prod_{cyc} (a_1 a_2^2 + a_2 + 1)$$

5. Given that x, y, z are non-negative real numbers, prove that

$$(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) \geq (xy + yz + zx)^3$$

6. Given that a, b, c are the sides of a triangle satisfying $2A^2 \geq P$ where A is area and P is perimeter, prove that

$$(a+b)(b+c)(c+a) \leq a^2 b^2 c^2$$

6 Solutions

6.1 Problem Solutions

1. *Solution.*

   We want to prove that
$$\prod_{k=1}^{n}(a_k+k)\le(n+1)^n$$
   Note that when $a_k = n-k+1$, the two sequences $a_1, a_2, \ldots a_n$ and $1, 2, \ldots n$ are oppositely ordered. Call this specific permutation of $1, 2, \ldots n$ the sequence $A_1, A_2, \ldots A_k$. So, we have that the $RHS = (A_1{+}1)(A_2{+}2)\cdots(A_n{+}n)$. Since $\{A\}$ and $1, 2, \ldots n$ are oppositely ordered, by Reverse Rearrangement we have that
$$(A_1+1)(A_2+2)\cdots(A_n+n)=(n+1)^n\ge(a_1+1)(a_2+2)\cdots(a_n+n)$$
   where $a_1, a_2, \ldots a_n$ is a permutation of $1, 2, \ldots n$. But this is exactly what we're trying to prove, so we're done. $\square$

2. *Solution.*

   We want to prove
$$\prod_{i=1}^{n}\prod_{j=1}^{n}(a_i+b_j)\ge\prod_{k=1}^{n}(a_k+b_k)^n$$
   We see that the
$$RHS=\underbrace{(a_1+b_1)(a_1+b_1)\cdots(a_1+b_1)}_{n\text{ times}}\underbrace{(a_2+b_2)(a_2+b_2)\cdots(a_2+b_2)}_{n\text{ times}}\cdots\underbrace{(a_n+b_n)(a_n+b_n)\cdots(a_n+b_n)}_{n\text{ times}}$$
   Thus by Reverse Rearrangement,
$$RHS\le\Big((a_1{+}b_1)(a_1{+}b_2)\cdots(a_1{+}b_n)\Big)\Big((a_2{+}b_1)(a_2{+}b_2)\cdots(a_2{+}b_n)\Big)\cdots\Big((a_n{+}b_1)(a_n{+}b_2)\cdots(a_n{+}b_n)\Big)$$
   But the RHS of the above inequality is just $\displaystyle\prod_{i=1}^{n}\prod_{j=1}^{n}(a_i+b_j)$ so we are done. $\square$

3. *Solution.*

   We want to prove that
$$(x^2-x+y)(y^2-y+x)\ge x^2y^2$$
   Consider the function $f(x) = x^2 - x$; its increasing when $x \ge 1$ so we can use Reverse Rearrangement. By Corollary 1b, we have that
$$(x+f(x))(y+f(y))\le(x+f(y))(y+f(x))$$
   Subbing the function back in, we see that
$$(x+x^2-x)(y+y^2-y)\le(x+y^2-y)(y+x^2-x)$$
   But simplifying gives
$$(x^2-x+y)(y^2-y+x)\ge x^2y^2$$
   which is exactly what we wanted. $\square$

4. *Solution.*

   Note that this problem is a generalization of the previous problem; we have n variables instead of 2 variables. We can solve it with a similar fashion to the previous problem: let $f(x) = x^2 - x$. Since $f(x)$ is increasing for $x \geq 1$, we can use Reverse Rearrangement. By Corollary 1b, we have that

   $$(a_1 + f(a_1))(a_2 + f(a_2)) \cdots (a_n + f(a_n)) \leq (a_1 + f(a_n))(a_2 + f(a_1)) \cdots (a_n + f(a_{n-1}))$$

   Subbing the function back in, our inequality turns into

   $$(a_1 + a_1^2 - a_1)(a_2 + a_2^2 - a_2) \cdots (a_n + a_n^2 - a_n) \leq (a_1 + a_n^2 - a_n)(a_2 + a_1^2 - a_1) \cdots (a_n + a_{n-1}^2 - a_{n-1})$$

   but this simplifies to the inequality we want to prove, so we're done. $\square$

5. *Solution.*

   We can first simplify the inequality:

   $$\begin{aligned} \ln ex^{ex} \ln ey^{ey} \ln ez^{ez} &\geq \ln ex^{ey} \ln ey^{ez} \ln ez^{ex} \\ (\ln x^{ex} + 1)(\ln y^{ey} + 1)(\ln z^{ez} + 1) &\geq (\ln x^{ey} + 1)(\ln y^{ez} + 1)(\ln z^{ex} + 1) \\ (ex \ln x + 1)(ey \ln y + 1)(ez \ln z + 1) &\geq (ey \ln x + 1)(ez \ln y + 1)(ex \ln z + 1) \end{aligned}$$

   Now let $f(x) = e \ln x$. Confirm that $f(x)$ is non-negative and increasing when $x \geq 1$. By Corollary 2b, we see that

   $$(xe \ln x + 1)(ye \ln y + 1)(ze \ln z + 1) \geq (ye \ln x + 1)(ze \ln y + 1)(xe \ln z + 1)$$

   but this is exactly what we wanted to prove. $\square$

6. *Solution.*

   First off, we can write $k^{a_1 - a_2} = \dfrac{k^{a_1}}{k^{a_2}}$ which turns the inequality into

   $$\prod_{cyc} \left(\frac{k^{a_1}}{k^{a_2}} + 1 \right) \geq 2^n$$

   Multiplying both sides by $k^{a_1 + a_2 + \cdots + a_n}$ gives

   $$\prod_{cyc} (k^{a_1} + k^{a_2}) \geq k^{a_1 + a_2 + \cdots + a_n} \cdot 2^n$$

   However, note that $2^n = (1+1)^n$, so

   $$k^{a_1 + a_2 + \cdots + a_n} \cdot 2^n = (k^{a_1} + k^{a_1})(k^{a_2} + k^{a_2}) \cdots (k^{a_n} + k^{a_n}) = \prod_{cyc} (k^{a_1} + k^{a_1})$$

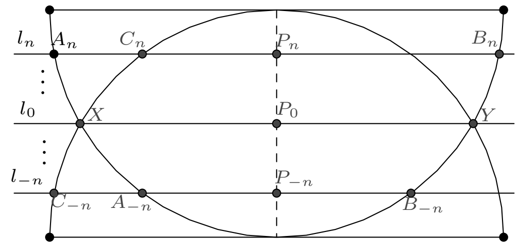
   Thus we want to prove that

   $$\prod_{cyc} (k^{a_1} + k^{a_2}) \geq \prod_{cyc} (k^{a_1} + k^{a_1})$$

   but this is true by Reverse Rearrangement so we're done. $\square$

7. *Solution.*

   We draw a vertical line directly through the middle of the diagram, and call the points of intersection of this line and line l_k as P_k for $k = -n \to n$.



We see that

$$\prod_{k=-n}^{n} |A_kB_k| = \prod_{k=-n}^{n} |A_kP_k| + |B_kP_k|$$

and that

$$\prod_{k=-n}^{n} |B_kC_k| = \prod_{k=-n}^{n} |B_kP_k| + |C_kP_k|$$

Clearly the sequences $|A_{-n}P_{-n}|, |A_{-n+1}P_{-n+1}|, \dots |A_nP_n|$ and $|B_{-n}P_{-n}|, |B_{-n+1}P_{-n+1}|, \dots |B_nP_n|$ are increasing.

Now, notice that

$$|A_kP_k| = |C_{-k}P_{-k}|$$

because lines l_k and l_{-k} are mirror images about line XY. Substituting that in, we just want to prove

$$\prod_{k=-n}^{n} |A_kP_k| + |B_kP_k| \leq \prod_{k=-n}^{n} |B_kP_k| + |A_{-k}P_{-k}|$$

By Reverse Rearrangement, this is true, so we are done. □

8. *Solution.*

We want to prove that

$$(a_1^2 + 2)(a_2^2 + 2) \cdots (a_n^2 + 2) \geq (a_1a_2 + 2)(a_2a_3 + 2) \cdots (a_na_1 + 2)$$

This seems like a straightforward application of Corollary 2b, except for the fact that we are adding 2 to each of the terms on the inside of the parentheses, instead of 1 as needed in the corollary. However, who said that the number had to be 1? If, when substituting $g(x) = \dfrac{1}{f(x)}$ in the proof of Corollary 2b, we had substituted $g(x) = \dfrac{2}{f(x)}$ instead, then our final result would have been

$$(a_1f(a_1) + 2)(a_2f(a_2) + 2) \cdots (a_nf(a_n) + 2) \geq (a_1f(a_2) + 2)(a_2f(a_3) + 2) \cdots (a_nf(a_1) + 2)$$

Now we can apply Corollary 2b straightforwardly, with $f(x) = x$:

$$(a_1 \cdot a_1 + 2)(a_2 \cdot a_2 + 2) \cdots (a_n \cdot a_n + 2) \geq (a_1 \cdot a_2 + 2)(a_2 \cdot a_3 + 2) \cdots (a_n \cdot a_1 + 2)$$

which is the inequality we wanted to prove. □

9. *Solution.*

We have a condition that $a_1a_2 \cdots a_n = 1$; thus we can use the substitution $a_k = \dfrac{b_k}{b_{k+1}}$ for $k = 1 \to n-1$ and $a_n = \dfrac{b_n}{b_1}$. This gives us a positive sequence $b_1, b_2, \dots b_n$ with no restrictions and the inequality

$$\left(\frac{b_1}{b_2} + k\right)\left(\frac{b_2}{b_3} + k\right) \cdots \left(\frac{b_n}{b_1} + k\right) \geq (k+1)^n$$

Multiplying both sides by $b_1 b_2 \cdots b_n$, we get

$$(b_1 + kb_2)(b_2 + kb_3) \cdots (b_n + kb_1) \geq (b_1 + kb_1)(b_2 + kb_2) \cdots (b_n + kb_n)$$

Now it is obvious how Reverse Rearrangement applies: let $f(x) = kx$. Confirm that $f(x)$ is increasing. Thus, using Corollary 1b, the inequality is true, so we are done. $\square$

10. *Solution.*

Let the point connected to A_k be $B_{\sigma(k)}$ for $k = 1 \to 5$. In addition, let $OA_k = a_k$ and $OB_k = b_k$. This makes

$$A_k B_{\sigma(k)} = \sqrt{a_k^2 + b_{\sigma(k)}^2}$$

Thus we want to find the maximum and minimum of

$$\prod_{k=1}^{5} \sqrt{a_k^2 + b_{\sigma(k)}^2} = \sqrt{\prod_{k=1}^{5} \left(a_k^2 + b_{\sigma(k)}^2\right)}$$

Note that by Reverse Rearrangement,

$$\prod_{k=1}^{5} \left(a_k^2 + b_k^2\right) \leq \prod_{k=1}^{5} \left(a_k^2 + b_{\sigma(k)}^2\right) \leq \prod_{k=1}^{5} \left(a_k^2 + b_{6-k}^2\right)$$

However,

$$\prod_{k=1}^{5} \left(a_k^2 + b_k^2\right) = (1^2 + 1^2)(2^2 + 2^2) \cdots (5^2 + 5^2) = 460800$$

and

$$\prod_{k=1}^{5} \left(a_k^2 + b_{6-k}^2\right) = (1^2 + 5^2)(2^2 + 4^2) \cdots (5^2 + 1^2) = 4867200$$

Thus,

$$\sqrt{460800} \leq \sqrt{\prod_{k=1}^{5} \left(a_k^2 + b_{\sigma(k)}^2\right)} \leq \sqrt{4867200}$$

which simplifies as

$$480\sqrt{2} \leq \sqrt{\prod_{k=1}^{5} \left(a_k^2 + b_{\sigma(k)}^2\right)} \leq 1560\sqrt{2}$$

which is what we wanted to prove. $\square$

6.2 Challenge Problem Solutions

1. *Solution.*

At first glance, this seems like a straightforward application of Reverse Rearrangement, but upon further inspection, we see that the trouble arises on the terms $a_1a_2, a_2a_3, \ldots a_na_1$. In order to successfully use Reverse Rearrangement, we have to make sure that the different variables are separated. To separate $(a_1a_2 + a_1 + a_2)$, we recognize the familiar factorization known as Simon's Favorite Factoring Trick:

$$a_1a_2 + a_1 + a_2 + 1 = (a_1 + 1)(a_2 + 1)$$

Thus, $a_1a_2 + a_1 + a_2 = (a_1 + 1)(a_2 + 1) - 1$. However, we still have the $a_1 + 1$ and $a_2 + 1$ terms together. To fix this, we can make the clever manipulation

$$(a_1 + 1)(a_2 + 1) - 1 = (a_2 + 1)\left(a_1 + 1 - \frac{1}{a_2 + 1}\right)$$

Why does this change things? Look at what happens when we plug this substitution back in:

$$\begin{aligned} RHS &= (a_1a_2 + a_1 + a_2)(a_2a_3 + a_2 + a_3)\cdots(a_na_1 + a_n + a_1) \\ &= \prod_{cyc}(a_1+1)(a_2+1) - 1 \\ &= \prod_{cyc}(a_2+1)\left(a_1 + 1 - \frac{1}{a_2+1}\right) \\ &= \left(\prod_{cyc} a_1 + 1\right)\left(\prod_{cyc} a_1 + 1 - \frac{1}{a_2+1}\right) \end{aligned}$$

But wait: following similar algebraic manipulations as before, we have

$$\begin{aligned} LHS &= (a_1^2 + 2a_1)(a_2^2 + 2a_2)\cdots(a_n^2 + 2a_n) \\ &= \prod_{cyc}(a_1+1)(a_1+1) - 1 \\ &= \prod_{cyc}(a_1+1)\left(a_1 + 1 - \frac{1}{a_1+1}\right) \\ &= \left(\prod_{cyc} a_1 + 1\right)\left(\prod_{cyc} a_1 + 1 - \frac{1}{a_1+1}\right) \end{aligned}$$

Thus we want to prove that

$$\left(\prod_{cyc} a_1 + 1\right)\left(\prod_{cyc} a_1 + 1 - \frac{1}{a_1+1}\right) \leq \left(\prod_{cyc} a_1 + 1\right)\left(\prod_{cyc} a_1 + 1 - \frac{1}{a_2+1}\right)$$

Immediately, we see the benefit of our substitution: we can cancel out $(a_1+1)(a_2+1)\cdots(a_n+1)$ from both sides to get

$$\left(\prod_{cyc} a_1 + 1 - \frac{1}{a_1+1}\right) \leq \left(\prod_{cyc} a_1 + 1 - \frac{1}{a_2+1}\right)$$

Now let $f(x) = 1 - \dfrac{1}{x+1}$. Confirm that $f(x)$ is an increasing function for $x \geq 0$. Now we can use Corollary 1b to get

$$(a_1 + f(a_1))(a_2 + f(a_2))\cdots(a_n + f(a_n)) \leq (a_1 + f(a_2))(a_2 + f(a_3))\cdots(a_n + f(a_1))$$

Substituting the definition of our function back in and we get

$$\prod_{cyc}\left(a_1 + 1 - \frac{1}{a_1+1}\right) \leq \prod_{cyc}\left(a_1 + 1 - \frac{1}{a_2+1}\right)$$

which is exactly what we wanted to prove, so we're done. $\square$

2. *Solution.*

First off, we notice that all the variables are in terms of $\sin\alpha, \sin\beta, \sin\gamma$ except $\sin 3\alpha$. Thus, let's change that using the triple angle formula:

$$3\sin\alpha - \sin 3\alpha = 3\sin\alpha - (3\sin\alpha - 4\sin^3\alpha) = 4\sin^3\alpha$$

The LHS is therefore equal to

$$\prod_{cyc}(4\sin^3\alpha) = 64\sin^3\alpha\sin^3\beta\sin^3\gamma$$

Thus we want to prove that

$$\prod_{cyc}(\sin^3\alpha + 3\sin\alpha\sin\beta\sin\gamma) \geq 64\sin^3\alpha\sin^3\beta\sin^3\gamma$$

We have an ugly $\sin\alpha\sin\beta\sin\gamma$ term that is hard to work with using the Reverse Rearrangement inequality, so let's first simplify by dividing both sides by $\sin\alpha\sin\beta\sin\gamma$:

$$\prod_{cyc}(\sin^2\alpha + 3\sin\beta\sin\gamma) \geq 64\sin^2\alpha\sin^2\beta\sin^2\gamma$$

However, we still have three variables per term on the LHS, which is hard to work with. The best way to decrease the number of variables per term is substitution, as we've seen from previous and will see on later problems, so let's go with that. However, how will we substitute?

Observe what happens when we multiply by $\sin\alpha\sin\beta\sin\gamma$ again but in a different way:

$$\prod_{cyc}(\sin^2\alpha\sin\beta + 3\sin^2\beta\sin\gamma) \geq 64\sin^3\alpha\sin^3\beta\sin^3\gamma$$

Suddenly, we see the substitution: let $a = \sin^2\alpha\sin\beta$, $b = \sin^2\beta\sin\gamma$, and $c = \sin^2\gamma\sin\alpha$. This turns the inequality into

$$\prod_{cyc}(a+3b) \geq 64abc$$

But by Reverse Rearrangement,

$$\prod_{cyc}(a+3b) \geq \prod_{cyc}(a+3a) = 64abc$$

so we are done. $\square$

3. *Solution.*

The LHS looks great for some Reverse Rearrangement, but the RHS not so much. We only have the product of two terms (and a constant).

But first things first, we have a condition that $xy + yz + zx = 1$; thus, lets homogenize. The LHS is simple:

$$\prod_{cyc}(x^2+y^2+2) = \prod_{cyc}(x^2+y^2+2xy+2yz+2zx)$$

But wait a second: $x^2+xy+yz+zx = (x+y)(x+z)$ and $y^2+xy+yz+zx = (y+z)(y+x)$ so we have

$$\prod_{cyc}(x^2+y^2+2xy+2yz+2zx) = \prod_{cyc}\big((x+y)(x+z)+(y+z)(y+x)\big) = \prod_{cyc}(x+y)(x+y+2z)$$

Now let's homogenize the RHS.

$$\begin{aligned}8(xyz-x-y-z)^2 &= 8\big(xyz-(xy+yz+zx)(x+y+z)\big)\\ &= 8\big(xyz-(3xyz+x^2y+x^2z+y^2x+y^2z+z^2x+z^2y)\big)^2\\ &= 8(x^2y+x^2z+y^2x+y^2z+z^2x+z^2y+2xyz)^2\\ &= 8\big((x+y)(y+z)(z+x)\big)^2\end{aligned}$$

Our inequality turns into

$$\prod_{cyc}(x+y)(x+y+2z) \geq 8\big((x+y)(y+z)(z+x)\big)^2$$

Dividing both sides by $(x+y)(y+z)(z+x)$, we get

$$\prod_{cyc}(x+y+2z) \geq 8(x+y)(y+z)(z+x)$$

We observe that we only have $x+y$, $y+z$, and $z+x$ terms on both sides of the inequality, which prompts us to substitute $a = x+y$, $b = x+z$, and $c = y+z$.

Thus we just need to prove the inequality

$$\prod_{cyc}(a+c) \geq 8abc$$

This is perfect for Reverse Rearrangement now:

$$\prod_{cyc}(a+c) \geq \prod_{cyc}(a+a) = (2a)(2b)(2c) = 8abc$$

so we are done. □

4. *Solution.*

The trouble about this problem is that both sides have multiple variables per term on it; in our previous problems, we usually see one side consist of only one variable per term. Since we don't have any expressions having one variable per term, why don't we create one? Looking at the LHS and RHS, it appears it is easier to work with the LHS, so we will do so.

We want to somehow transform

$$\prod_{cyc}(a_1^3 + a_2 + 1)$$

To do this, we think about using Reverse Rearrangement; after all, that's what it's good at: permutating variables.

Let $f(x) = x^3 + 1$. Check that this is increasing. Applying Corollary 1b, we get that

$$\begin{aligned}\prod_{cyc}(a_1^3 + a_2 + 1) &= \prod_{cyc}(a_2 + f(a_1)) \\ &\geq \prod_{cyc}(a_1 + f(a_1)) \\ &= \prod_{cyc}(a_1^3 + a_1 + 1)\end{aligned}$$

Thus, we just need to prove that

$$\prod_{cyc}(a_1^3 + a_1 + 1) \geq \prod_{cyc}(a_1 a_2^2 + a_2 + 1)$$

Reverse Rearrangement worked for us once. Why not use it again? However, this time it cannot be directly applied since we have a nasty $a_1 a_2^2$ term that we need to separate.

In our attempts to simplify it:

$$\begin{aligned}\prod_{cyc}(a_1a_2^2+a_2+1) &= \prod_{cyc} a_2^2\left(a_1+\frac{1}{a_2}\right)+1\\ &= \prod_{cyc} a_2^2\left(a_1+\frac{1}{a_2}+\frac{1}{a_2^2}\right)\end{aligned}$$

To get our inequality to have the right format for the permutation needed for Reverse Rearrangement, we do the same algebraic manipulations for $\prod_{cyc}(a_1^3+a_1+1)$:

$$\begin{aligned}\prod_{cyc}(a_1^3+a_1+1) &= \prod_{cyc} a_1^2\left(a_1+\frac{1}{a_1}\right)+1\\ &= \prod_{cyc} a_1^2\left(a_1+\frac{1}{a_1}+\frac{1}{a_1^2}\right)\end{aligned}$$

We simplified our inequality to the following:

$$\prod_{cyc} a_1^2\left(a_1+\frac{1}{a_1}+\frac{1}{a_1^2}\right) \geq \prod_{cyc} a_2^2\left(a_1+\frac{1}{a_2}+\frac{1}{a_2^2}\right)$$

Dividing both sides by $a_1^2a_2^2\cdots a_n^2$:

$$\prod_{cyc}\left(a_1+\frac{1}{a_1}+\frac{1}{a_1^2}\right) \geq \prod_{cyc}\left(a_1+\frac{1}{a_2}+\frac{1}{a_2^2}\right)$$

Now this looks more like an inequality we can use Reverse Rearrangement on. Let $f(x)=\frac{1}{x}+\frac{1}{x^2}$. Confirm that $f(x)$ is decreasing. Thus, we can use Corollary 1a to get:

$$\begin{aligned}\prod_{cyc}\left(a_1+\frac{1}{a_1}+\frac{1}{a_1^2}\right) &= \prod_{cyc}(a_1+f(a_1))\\ &\geq \prod_{cyc}(a_1+f(a_2))\\ &= \prod_{cyc}\left(a_1+\frac{1}{a_2}+\frac{1}{a_2^2}\right)\end{aligned}$$

And we are done. $\square$

5. *Solution.*

There doesn't seem to be any straightforward way to solve this problem, mainly because the RHS has all three variables involved in each term. We don't see an easy way to permutate a variable of a term of the LHS to form a term of the RHS mainly because there is both an x^2 and a y^2 term in each of the terms, so we need to find some way to simplify one or both sides.

The first thing that comes to mind is multiplying both sides by $(x-y)(y-z)(z-x)$ because it turns the $LHS=(x^3-y^3)(y^3-z^3)(z^3-x^3)$, but we cannot do this reliably because $(x-y)(y-z)(z-x)$ may be negative.

In an effort to simplify the LHS by using substitution, we divide both sides by $x^2y^2z^2$:

$$\left(1+\frac{y}{x}+\frac{y^2}{x^2}\right)\left(1+\frac{z}{y}+\frac{z^2}{y^2}\right)\left(1+\frac{x}{z}+\frac{x^2}{z^2}\right) \geq \frac{(xy+yz+zx)^3}{x^2y^2z^2}$$

Let's substitute $a = \dfrac{y}{x}$, $b = \dfrac{z}{y}$, and $c = \dfrac{x}{z}$. Note the implied condition that $abc = 1$.

$$(1+a+a^2)(1+b+b^2)(1+c+c^2) \geq \frac{(xy+yz+zx)^3}{x^2y^2z^2}$$

In lieu of completing the substitution, we transform the RHS to be in a form that we can substitute our a, b, c in:

$$\begin{aligned}
\frac{(xy+yz+zx)^3}{x^2y^2z^2} &= \left(\frac{xy+yz+zx}{x^2}\right)\left(\frac{xy+yz+zx}{y^2}\right)\left(\frac{xy+yz+zx}{z^2}\right) \qquad (*)\\
&= \left(\frac{y}{x}+\frac{yz}{x^2}+\frac{z}{x}\right)\left(\frac{z}{y}+\frac{zx}{y^2}+\frac{x}{y}\right)\left(\frac{x}{z}+\frac{xy}{z^2}+\frac{y}{z}\right)\\
&= \left(a+\frac{a}{c}+\frac{1}{c}\right)\left(b+\frac{b}{a}+\frac{1}{a}\right)\left(c+\frac{c}{b}+\frac{1}{b}\right)
\end{aligned}$$

To make the degrees of each term of $1+a+a^2$ and $a+\dfrac{a}{c}+\dfrac{1}{c}$ the same pairwise, we multiply $\dfrac{1}{c}$ by abc (which remember equals 1). This is an important step for using the Reverse Rearrangement; usually, the degrees of each of the terms pairwise needs to be the same. This gives us

$$\left(a+\frac{a}{c}+\frac{1}{c}\right)\left(b+\frac{b}{a}+\frac{1}{a}\right)\left(c+\frac{c}{b}+\frac{1}{b}\right) = \left(a+\frac{a}{c}+ab\right)\left(b+\frac{b}{a}+bc\right)\left(c+\frac{c}{b}+ca\right)$$

Uh oh. We still have three variables in the RHS, and no clear way to permutate the variables. Let's go back to $(*)$ and try grouping the division of $x^2y^2z^2$ differently, this time hopefully getting only two variables per term on the RHS.

$$\begin{aligned}
\frac{(xy+yz+zx)^3}{x^2y^2z^2} &= \left(\frac{xy+yz+zx}{xy}\right)\left(\frac{xy+yz+zx}{yz}\right)\left(\frac{xy+yz+zx}{zx}\right)\\
&= \left(1+\frac{z}{x}+\frac{z}{y}\right)\left(1+\frac{x}{y}+\frac{x}{z}\right)\left(1+\frac{y}{z}+\frac{y}{x}\right)\\
&= \left(1+\frac{1}{c}+b\right)\left(1+\frac{1}{a}+c\right)\left(1+\frac{1}{b}+a\right)
\end{aligned}$$

Just like last time, we multiply $\dfrac{1}{a}$ by abc to make it the same degree pairwise as a^2:

$$\left(1+\frac{1}{c}+b\right)\left(1+\frac{1}{a}+c\right)\left(1+\frac{1}{b}+a\right) = (1+ab+b)(1+bc+c)(1+ca+a)$$

Aha! This time, we have only two variables per term in the expression. We have now reduced the problem to proving that given $abc = 1$,

$$(1+a+a^2)(1+b+b^2)(1+c+c^2) \geq (1+ab+b)(1+bc+c)(1+ca+a)$$

Now we proceed typically: we need to separate the variables in the RHS so we can successfully turn it into the LHS through a permutation. We see that $1+ab+b = 1+(a+1)b$ and that $1+b+b^2 = 1+(b+1)b$. Now it is clear what to do:

Let $f(x) = x+1$. This is an increasing function, so using Corollary 2b, we have that

$$(af(a)+1)(bf(b)+1)(cf(c)+1) \geq (af(c)+1)(bf(a)+1)(cf(b)+1)$$

Subbing in the definition of $f(x)$, we get that

$$(a(a+1)+1)(b(b+1)+1)(c(c+1)+1) \geq (a(c+1)+1)(b(a+1)+1)(c(b+1)+1)$$

which simplifies into

$$(a^2+a+1)(b^2+b+1)(c^2+c+1) \geq (ab+b+1)(bc+c+1)(ca+a+1)$$

But this is exactly what we wanted to prove. Thus, we are finally done! $\square$

6. *Solution.*

The most ugliest part about this problem is the condition $2A^2 \geq P$; it's really hard to work with things like these. Thus, let's first try to simplify it by representing both things in terms of the sides a, b, c.

We know that $P = a+b+c$, and $A = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$ so our condition turns into

$$\begin{aligned} 2\left(\frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}\right)^2 &\geq a+b+c \\ \frac{1}{8}(a+b+c)(a+b-c)(a-b+c)(-a+b+c) &\geq a+b+c \\ (a+b-c)(a-b+c)(-a+b+c) &\geq 8 \end{aligned}$$

We now have the condition in terms of a, b, c, but it still looks pretty ugly. To simplify matters, we use the Ravi substitution: let $a = x+y$, $b = y+z$, and $c = z+x$.

This turns the condition into

$$(x+y+y+z-z-x)(y+z+z+x-x-y)(z+x+x+y-y-z) \geq 8$$

which simplifies nicely into

$$xyz \geq 1$$

Of course, we also need to substitute into the actual inequality now:

$$(x+2y+z)(y+2z+x)(z+2x+y) \leq (x+y)^2(y+z)^2(z+x)^2$$

Let's expand each of the binomials on the RHS:

$$\prod_{cyc}(x+2y+z) \leq \prod_{cyc}(x^2+2xy+y^2)$$

Hmm... The coefficients of the variables look surprisingly similar, with only the degrees of the two sides differing. This tells us to homogenize. The first thing that comes to mind is multiplying the LHS by xyz, which is legal because $1 \cdot LHS \leq xyz \cdot LHS$ which makes the inequality stronger. However, this gives

$$\prod_{cyc}(x^2+2xy+xz) \leq \prod_{cyc}(x^2+2xy+y^2)$$

which looks kind of like it can be solved using Reverse Rearrangement except for the fact that we need to change $xz \to y^2$ which is challenging.

But since we already are this close, let's see if we can get any closer by multiplying or dividing both sides by xyz. Multiplying by xyz will just make the degree higher, which might work but will be a pain, so we divide by xyz on both sides. Also, something tells us that in order to make this work out, we'll need some sort of substitution, just like how we solved Challenge problem 5, which also had all three variables in each term.

Divide once:

$$\frac{1}{xyz}\prod_{cyc}(x^2+2xy+xz) \le \frac{1}{xyz}\prod_{cyc}(x^2+2xy+y^2)$$
$$\prod_{cyc}(x+2y+z) \le \prod_{cyc}\left(x+2y+\frac{y^2}{x}\right)$$

That $\frac{y^2}{x}$ term looks real ugly, and we still don't see an easy substitution.

Divide twice:

$$\frac{1}{xyz}\prod_{cyc}(x+2y+z) \le \frac{1}{xyz}\prod_{cyc}\left(x+2y+\frac{y^2}{x}\right)$$
$$\prod_{cyc}\left(\frac{x}{y}+2+\frac{z}{y}\right) \le \prod_{cyc}\left(\frac{x}{y}+2+\frac{y}{x}\right)$$

Suddenly the substitution becomes as clear as day. Let $a' = \frac{x}{y}$, $b' = \frac{y}{z}$, and $c' = \frac{z}{x}$.

The inequality becomes

$$\prod_{cyc}\left(a'+2+\frac{1}{b'}\right) \le \prod_{cyc}\left(a'+2+\frac{1}{a'}\right)$$

Now the application of Reverse Rearrangement is obvious: let $f(x) = 2+\frac{1}{x}$; confirm that it is decreasing for $x > 0$. By Corollary 1a, we have that

$$\prod_{cyc}\left(a'+2+\frac{1}{a'}\right) \ge \prod_{cyc}\left(a'+2+\frac{1}{b'}\right)$$

which is exactly our inequality, so we are done. $\square$